Synthetic Control Methods by Density Matching under Implicit Endogeneity

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1. ABSTRACT

- Synthetic Control Methods (SCMs; Abadie 2002).
- **Idea**: estimate counterfactual outcomes for a treated unit by a weighted sum of observed outcomes from untreated units.
- We find an **implicit endogeneity** problem in SCMs.

→ Least squares is inappropriate for estimating weights in SCMs.

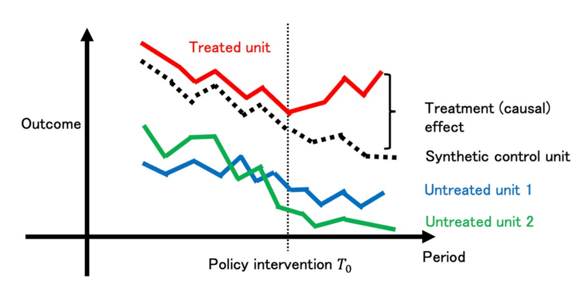
- We propose SCMs by density matching.

2. PROBLEM SETTING

- $J+1$ units, $j \in \mathcal{J} := \{0,1,2,\dots,J\}$.
- $j=0$: **Treated unit** (a unit affected by the policy intervention).
- $j \in \mathcal{J}^U : \mathcal{J} \setminus \{0\}$: **Untreated units**.
- T Periods, $t \in \mathcal{T} := \{1,2,\dots,T\}$.
- Intervention occurs at $t = T_0 < T$.
- $t \in \mathcal{T}_0 := \{1,2,\dots,T_0\}$: before the intervention.
- $t \in \mathcal{T} \setminus \mathcal{T}_0$: after the intervention ($T_1 := |\mathcal{T}_1| = T - T_0$).
- **Potential outcomes** (Neyman, 1923; Rubin, 1974):
- For each unit $j \in \mathcal{J}$ and period $t \in \mathcal{T}$, define $\left(Y^I_{j,t}, Y^N_{j,t}\right) \in \mathbb{R}^2$.
- Y^I_t and Y^N_t are outcomes with and without interventions.
- $\mathbb{E}_{j,t}$: expectation over Y^I_t and Y^N_t.
- **Observations**:
- Observe one of the outcomes corresponding to intervention.
- We observe $Y_{j,t} \in \mathbb{R}$ such that

$$Y_{0,t} = \begin{cases} Y^I_{0,t} & if \;\; t \in \mathcal{T}_1 \\ Y^N_{0,t} & if \;\; t \in \mathcal{T}_0 \end{cases}, \qquad Y_{j,t} = Y^N_{j,t} \quad for\; j \in \mathcal{J}^U.$$

- Causal effects:
- $\tau_{0,t} := \mathbb{E}_{0,t}\left[Y^I_{0,t} - Y^N_{0,t}\right] \quad for\; t \in \mathcal{T}_1.$



- Estimating the causal effect by predicting $Y^N_{0,t}$ for $t \in \mathcal{T}_1$.

3. IMPLICIT ENDOGENEITY

- **Assumption**: there is a set of weights $w^* = \left(w^*_j\right)$ such that

$$\mathbb{E}_{0,t}\left[Y^N_{0,t}\right] = \sum_{j \in \mathcal{J}^U} w^*_j \,\mathbb{E}_{j,t}\left[Y^N_{j,t}\right] \qquad \sum_{j \in \mathcal{J}^U} w^*_j = 1.$$

- Let $Y^N_{j,t} = \mathbb{E}_{j,t}\left[Y^N_{j,t}\right] + \varepsilon_{j,t}$. Under this assumption, it holds that

$$Y^N_{0,t} = \sum_{j \in \mathcal{J}^U} w^*_j Y^N_{j,t} - \sum_{j \in \mathcal{J}^U} w^*_j \varepsilon_{j,t} + \varepsilon_{0,t} = \sum_{j \in \mathcal{J}^U} w^*_j Y^N_{j,t} + v_t.$$

correlation

- Implicit endogeneity: correlation between $Y^N_{j,t}$ and v_t.
- Least-squares estimators $\widehat{w}^{\mathrm{LS}}_j$ are biased: $\widehat{w}^{\mathrm{LS}}_j \xrightarrow{p} \widetilde{w}_j \neq w^*_j$.

4. SCM BY DENSITY MATCHING

- Consider another estimation strategy.
- Assume **mixture models** and estimate the weights by generalized method of moments (**GMM**).
- $p_{j,t}(y)$: density of $Y^N_{j,t}$
- Mixture models between $p_{0,t}(y)$ and $\{p_{j,t}(y)\}_{j \in \mathcal{J}^U}$:

$$p_{0,t}(y) = \sum_{j \in \mathcal{J}^U} w^*_j \, p_{j,t}(y).$$

- GMM:
- Moment conditions: $\mathbb{E}_{0,t}\left[\left(Y^N_{0,t}\right)^\gamma\right] = \sum_{j \in \mathcal{J}^U} w^*_j \,\mathbb{E}_{j,t}\left[\left(Y^N_{j,t}\right)^\gamma\right] \;\; \forall \gamma \in \mathbb{R}^+$.
- Empirical moment conditions:

$$\widehat{m}_\gamma(w) := \frac{1}{T_0} \sum_{t \in \mathcal{T}_0} \left\{ \left(Y^N_{0,t}\right)^\gamma - \sum_{j \in \mathcal{J}^U} w_j \left(Y^N_{j,t}\right)^\gamma \right\}.$$

- A set of positive values Γ, e.x., $\Gamma = \{1,2,3,4,5\}$.
- Estimate w^*_j by

$$w^{\mathrm{GMM}} := \left(w^{\mathrm{GMM}}_j\right)_{j \in \mathcal{J}^U} = \arg \min_{(w_j): \sum_{j \in \mathcal{J}^U} w_j = 1} \sum_{\gamma \in \Gamma} \left(\widehat{m}_\gamma(w)\right)^2.$$

- Asymptotically unbiased: $w^{\mathrm{GMM}}_j \xrightarrow{p} w^*_j$.

6. FINE-GRAINED MODELS

- **Linear factor models** are usually assumed in SCMs:

$$Y^N_{j,t} = c_j + \delta_t + \lambda_t \mu_j + \varepsilon_{j,t}, \qquad Y^I_{j,t} = \tau_{0,t} + Y^N_{j,t}$$

- Mixture models imply factor models under some assumptions.
- **Fine-grained models** (Shi et al., 2021).
- Assume that $Y^N_{j,t}$ represents a group-level outcome.

= We In each unit, there are unobserved units that constitute $Y^N_{j,t}$.

→ Under some assumptions, $p_{0,t}(y) = \sum_{j \in \mathcal{J}^U} w^*_j \, p_{j,t}(y)$ holds, and $p_{j,t}(y)$ is compatible to linear factor models.

5. INFERENCE

- Consider hypothesis testing for $H_0: \tau_{0,t} = 0$ for $t \in \mathcal{T}_1$.
- Note that $\tau_{0,t} = Y^I_{0,t} - Y^N_{0,t}$ under the linear factor model.
- Sample size may not be sufficient in $\mathcal{T}_1$, e.x., $T_1 = 1$.

→ **Conformal inference**.

- This method returns confidence intervals for finite samples.

7. EXPERIMENTS

- Let $\Gamma = \{1,2,\dots,G-1,G\}$.

G is chosen from $\{2,3,5,10,20,30,40,50,60,70,80,90,100\}$.

- **Simulation studies.**
- Generate $Y_{j,t}$ from gaussian distributions.

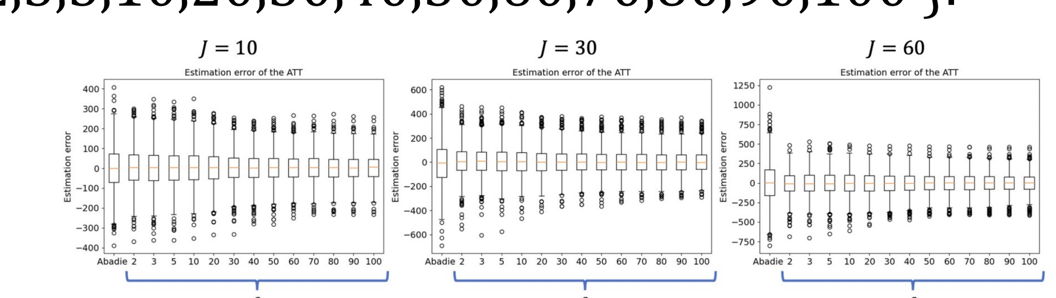
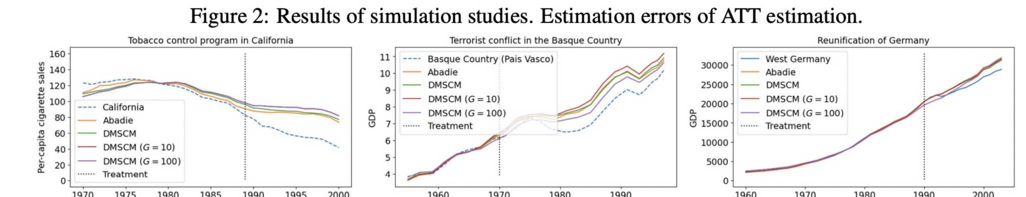


Figure 2: Results of simulation studies. Estimation errors of ATT estimation.

- **Empirical studies.**
- Conduct empirical studies using datasets used in existing studies.



Reference
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